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REPRESENTATION OF MEROTOPIC AND NEARNESS SPACES*

Wolfgang WEISS
FB Mathematik, TH Darmstadt, Schlossgartenstr. 7, D-6100 Darmstadt, Fed. Rep. Germany

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The category of merotopic spaces and uniformly continuous functions is shown to be adjoint to a category of completely distributive lattices with distinguished bases and grills and complete homomorphisms preserving these structure-sets. Suitable (co-)restrictions yield a lattice-theoretical representation of nearness spaces.

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nearness space   completely distributive lattice
scale   base   grill   initial mono-source

1. Introduction

This note is devoted to a representation theory of merotopic and nearness spaces by means of completely distributive lattices. The guiding idea lies in the attempt to translate classical duality theories of lattices and topological spaces (e.g. [1, 2]) into the context of uniform structures, i.e. (pre-)merotopic, nearness and uniform spaces. Since these structures consist of families of subsets of the underlying set, it is necessary to investigate lattice-theoretical properties of collections of families of subsets. The basic observation that the collection of all near families is determined by its set of near stacks leads to the notion of the scale of a set, which consists of all stacks (Definition 2.1). It turns out that this is a completely distributive lattice in which every element is the limsup of principal ultrafilters (Proposition 2.2) and this property motivates the study of base-lattices (Definition 2.3 and Theorem 2.5).

For every merotopic space the distinguished set of near stacks is a lattice-grill (i.e. the complement of an ideal) within the scale (Lemma 3.2). The principal ultrafilters form a base of the scale and enable us to recover the original space. More generally, any pair of a base-lattice and a lattice-grill ('grill-lattice', Definition

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3.3) can be endowed with a merotopic structure by means of a process, which is very similar to the canonical cluster-completion of nearness spaces (Proposition 3.7, see [4, 5.7, 5.8]). These assignments can be extended to functors and we obtain an adjunction with a natural isomorphism as its unit (Theorem 3.9). A suitable (co-)restriction of this adjunction yields a lattice-theoretical characterization of nearness spaces (Theorem 4.7). Moreover, nice categorical features (existence of initial lifts for monosources, preservation of initial sources) are established (Theorem 3.11 and Proposition 3.12).

With respect to the theory of nearness spaces we shall use the following terminology and notation: For any set X let $\mathcal{P}X$ denote the powerset of X and $\mathcal{P}^2X=\mathcal{P}\mathcal{P}X$. If $\mathcal{A}$ and $\mathcal{B}$ are families of subsets of X, one says that $\mathcal{A}$ *corefines* $\mathcal{B}$ ($\mathcal{A}>\mathcal{B}$) if for every $A\in\mathcal{A}$ there exists some $B\in\mathcal{B}$ with $A\supset B$.

The family $\mathcal{A}$ is a *stack* if $\mathcal{A}=\operatorname{stack}(\mathcal{A})=\{C\subset X\mid \exists A\in\mathcal{A}\colon A\subset C\}$. $\sigma(X)$ denotes the set of all stacks of X. One immediately observes that $\mathcal{A}>\mathcal{B}$ if and only if $\mathcal{A}\subset\operatorname{stack}(\mathcal{B})$. If $\xi\subset\mathcal{P}^2X$, $\mathcal{A}\subset\mathcal{P}X$ and $A\subset X$ let $\operatorname{cl}_\xi A=\{x\in X\mid\{\{x\},A\}\in\xi\}$ and $\operatorname{cl}_\xi\mathcal{A}=\{\operatorname{cl}_\xi A\mid A\in\mathcal{A}\}$. The principal filter generated by A is denoted by $\bar{A}$ and for every $x\in X$ let $\dot{x}=\overline{\{x\}}$. Consider the following conditions for a collection $\xi\subset\mathcal{P}^2X$.

(M1) $\mathcal{A}>\mathcal{B}\in\xi\Rightarrow\mathcal{A}\in\xi$,

(M2) $\emptyset\in\xi$, $\mathcal{P}X\notin\xi$,

(M3) $\bigcap\mathcal{A}\neq\emptyset\Rightarrow\mathcal{A}\in\xi$,

(M4) $\mathcal{A}\vee\mathcal{B}=\{A\cup B\mid A\in\mathcal{A}, B\in\mathcal{B}\}\in\xi\Rightarrow\mathcal{A}\in\xi$ or $\mathcal{B}\in\xi$,

(N) $\operatorname{cl}_\xi\mathcal{A}\in\xi\Rightarrow\mathcal{A}\in\xi$.

The pair (X,ξ) is called a *(pre-)merotopic space* if it satisfies (M1)-(M4) ((M1)-(M3)), and *nearness space* if it is a merotopic space satisfying (N). The members of ξ are called *near* families. A map $f\colon(X,\xi)\to(Y,\eta)$ between premerotopic spaces is *uniformly continuous* if $f(\xi)\subset\eta$. Denote the resulting categories by **PMer**, **Mer** and **Near** and the obvious forgetful functor by $V\colon\mathbf{PMer}\to\mathbf{Set}$. With respect to the theory of merotopic and nearness spaces the original papers of Katětov [7] and Herrlich [4] and the survey article [5] are excellent sources.

Terminology of categorical statements will be in accordance with [6] for general background, [12] for categorical algebra and [5] for categorical topology.

Let L be a completely distributive lattice, B, $G\subset L$ and $\mathcal{A}\subset\mathcal{P}L$. The *limes superior* of $\mathcal{A}$ is $\operatorname{limsup}\mathcal{A}=\inf\{\sup A\mid A\in\mathcal{A}\}$. B is a *base* of L if every element of L is the limes superior of subsets of B, and G is a *lattice-grill* if (i) $G\neq L$ and (ii) for all $x,y\in L$: $x\vee y\in G\Leftrightarrow x\in G$ or $y\in G$. An element x of L is $\bigwedge$*-prime* ($\wedge$*-prime*) if for every (finite) subset $M\subset L$:

$$x\geqslant\inf M\ \Rightarrow\ \exists m\in M\quad\text{with } x\geqslant m.$$

The dual notions are $\bigvee$*-prime* and $\vee$*-prime*. The set $P(L)$ of *principal elements* consists of all elements, which are both $\bigwedge$-prime and $\bigvee$-prime.

A map $f\colon L\to M$ between complete lattices is a *complete homomorphism* if it preserves arbitrary inf's and sup's. In particular $f(0)=0$ and $f(1)=1$. The category of completely distributive lattices and complete homomorphisms is denoted by **CDL**.

2. Base-lattices

The notion of a base-lattice emerges from the following observations:

2.1. Definition. Let X be a set. Then $S(X) = (\sigma(X), \leqslant)$ with $\mathcal{A} \leqslant \mathcal{B}$ if and only if $\mathcal{A} \supset \mathcal{B}$ is the *scale* of X.

2.2. Proposition. *Let X be a set, $S = S(X)$ its scale, $\mathcal{A}, \mathcal{B} \in S$ and $\omega \subset S$.*

(1) *S is completely distributive lattice with $\mathcal{P}X = 0_S \neq 1_S = \emptyset$ and $\inf_S \omega = \bigcup \omega$, $\sup_S \omega = \bigcap \omega$.*

(2) $\mathcal{A} \vee \mathcal{B} = \mathcal{A} \cap \mathcal{B} = \sup_S\{\mathcal{A}, \mathcal{B}\}$.

(3) *The principal filters $\bar{A}$ $(A \subset X)$ coincide with the $\bigwedge$-prime elements of S.*

(4) *The grills of X are precisely the $\vee$-prime elements of S.*

(5) *The principal ultrafilters are the principal elements of S, i.e. $P(S) = \{\dot{x} \mid x \in X\}$.*

(6) *Every stack $\mathcal{A} \in S$ is the limes superior of principal ultrafilters*: $\mathcal{A} = \inf\{\sup\{\dot{x} \mid x \in A\} \mid A \in \mathcal{A}\}$.

The proof is straightforward and therefore omitted. Properties (1), (5) and (6) motivate the following definition.

2.3. Definition. A pair (L, B) consisting of a completely distributive lattice L and a base $B \subset L$ is a *base-lattice.* Complete homomorphisms between base-lattices preserving the distinguished bases are called *base-homomorphisms* and the resulting category is denoted by **BL**.

'Par abus de langage' base-lattices (L, B) are sometimes denoted by L. If $x \in L$, let $\mathcal{A}_x = \{A \subset B \mid \sup A \geqslant x\}$.

Obviously every scale is a base-lattice taking the set of principal elements as a (minimal) base. Moreover, in every completely distributive lattice each element is the inf of $\wedge$-primes [2, I.3.15], i.e. the set of $\wedge$-primes is a base. The category **BL** is one of the categories proved to be universal, i.e. such that each concrete category can be fully embedded into it (see [11, V.4.23], where one can find also further references).

2.4. Definition. (1) The assignments $X \to SX = (S(X), P(S(X)))$ and $(f: X \to Y) \to (Sf: SX \to SY, \mathcal{A} \to \{E \subset Y \mid f^{-1}[E] \in \mathcal{A}\})$ define the *scale-functor* S: **Set** $\to$ **BL**.

(2) The *base-functor* B: **BL** $\to$ **Set** is defined by $(L, B) \to BL = B$ and $(f: (L, B) \to (K, C)) \to Bf = f|_B : B \to C$.

Since one always has $Sf(\dot{x}) = \dot{\overline{f(x)}}$, S is in fact a well-defined functor. The following theorem is contained in section V of [11].

2.5. Theorem. *The scale-functor* $S:\mathbf{Set}\to\mathbf{BL}$ *is left adjoint to the base-functor* $B:\mathbf{BL}\to\mathbf{Set}$. *The unit* η *of this adjunction is a natural isomorphism and is determined by*

$$\eta_X:\ X\to BSX,\quad x\to\dot{x}.$$

The co-unit ε *is given by*

$$\varepsilon_L:\ SBL\to L,\quad \mathscr{A}\to\operatorname{limsup}\mathscr{A}.$$

Proof. In order to show that η_X is a B-universal map, consider the following diagram:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "f"'] & BSX \arrow[d, dashed, "B\bar{f}"] & SX \arrow[d, dashed, "\bar{f}"] \\ & BK & K \end{tikzcd}$$

Determine a map $\bar{f}:SX\to K$ according to these rules:

$$\bar{f}:\ \begin{cases}\dot{x}\to f(x),\\ \bar{A}\to\sup(f[A]),\\ \mathscr{A}\to\operatorname{limsup}(f\mathscr{A})=\inf\{\sup(f[A])\mid A\in\mathscr{A}\}.\end{cases}$$

Some elementary calculations show that $\bar{f}$ is a complete homomorphism. Obviously it preserves also bases, i.e. $\bar{f}$ is the unique **BL**-morphism making the above diagram commutative. Since the co-unit $\varepsilon=(\varepsilon_L)$ is already determined by the request to satisfy

$$B\varepsilon_L\cdot\eta_{BL}=1_{BL}$$

[6, 26.11], this construction yields

$$\varepsilon_L(\mathscr{A})=\overline{1_{BL}}(\mathscr{A})=\operatorname{limsup}\mathscr{A}.\qquad\square$$

Obviously the conclusions of this theorem remain true if one replaces **BL** by any full subcategory containing all scales. We shall use such restricted adjunction in the next section. The special choice of the base of scale-lattices is necessary in order to ensure $\bar{f}$ to be a base-preserving map. However, in case one is only interested in base-lattices with trivial bases, i.e. pairs (L, L), one could assign to every scale $S(X)$ the trivial base $S(X)$ and the above theorem is still valid. These remarks provide a proof for the fact that the category **CDL** has free objects, which are (isomorphic to) scale-lattices [10].

Identifying the elements of the base of a base-lattice L with base-homomorphisms from the scale-lattice of a singleton set $1=\{0\}$ into L one immediately obtains that the base-functor is represented by $S1$, i.e. $B\simeq\hom_{\mathbf{BL}}(S1,-)$. Moreover, the scale-lattice of any nonempty set is a separator in **BL**. In particular B is a faithful functor. If one considers a self-map of a four-element-chain reversing the non-extremal elements, one has to conclude that the base-functor is not full.

2.6. Definition. Let $¶ = (BS, \eta, B\varepsilon S)$ denote the associated monad of the adjunction $S \dashv B$, $\mathbf{Set}^{¶}$ the corresponding Eilenberg–Moore-category and $K : \mathbf{BL} \to \mathbf{Set}^{¶}$ the canonical comparison functor.

2.7. Proposition. (1) $\mathbf{Set}^{¶} \simeq \mathbf{Set}$.
(2) $(K : \mathbf{BL} \to \mathbf{Set}^{¶}) \simeq (B : \mathbf{BL} \to \mathbf{Set})$.

Proof. The ¶-algebras are pairs of sets and structure-morphisms $(X, h_X : BSX \to X, \dot{x} \to x)$. The ¶-morphisms are maps $f: X \to Y$ satisfying

$$\begin{array}{ccc} BSX & \xrightarrow{BSf} & BSY \\ {\scriptstyle h_X}\downarrow & & \downarrow{\scriptstyle h_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

This diagram is automatically satisfied, i.e. it does not restrict the set of morphisms. Therefore one can identify $\mathbf{Set}^{¶}$ and $\mathbf{Set}$ trivially and under this identification the comparison functor is (isomorphic to) the base-functor. □

Applying the characterization theorem of Felscher for algebraic categories over **Set** [12, II.3.14] and combining it with the fact that the base-functor is not full one observes that **BL** is not an algebraic category.

3. Grill-lattices

3.1. Definition. If (X, ξ) is a premerotopic space,

$$\tau(\xi) = \xi \cap S(X)$$

is called the *tribe* of (X, ξ).

3.2. Lemma
(1) *Let (X, ξ) be a (pre-)merotopic space.*
 (i) *$SX = (S(X), P(S(X)))$ is a base-lattice.*
 (ii) *$\emptyset \neq \tau(\xi) \neq S(X)$ is lattice-grill (upper set).*
 (iii) *$P(S(X)) \subset \tau(\xi)$.*
(2) *A map $f: (X, \xi) \to (Y, \eta)$ between premerotopic spaces is uniformly continuous if and only if*

$$Sf[\tau(\xi)] \subset \tau(\eta).$$

Proof. (1): Clear by virtue of Proposition 2.2.
(2): f uniformly continuous iff $\forall \mathscr{A} \in \xi \; f\mathscr{A} > f(\operatorname{stack}(\mathscr{A})) > f\mathscr{A} \in \eta$ iff $\forall \mathscr{A} \in \tau(\xi) \; Sf(\mathscr{A}) > f\mathscr{A} > Sf(\mathscr{A}) \in \tau(\eta)$. □

These observations motivate the following notions:

3.3. Definition

(1) A pair (L, G) is called *(pre-)grill-lattice* provided

(i) L is a base-lattice,

(ii) $\emptyset \neq G \neq L$ is lattice-grill (upper set) and

(iii) $BL \subset G$.

(2) A map $f:(L, G) \to (K, H)$ between pregrill-lattices is *grill-continuous* if and only if

(i) $f: L \to K$ is base-homomorphism and

(ii) $f[G] \subset H$.

(3) The category of (pre-)grill-lattices and grill-continuous homomorphisms is denoted by **GrL** (**PGrL**).

According to (ii) of Definition 3.3(1) the top and bottom elements of pregrill-lattices are always different. Since $0 = \sup \emptyset$, it is no loss of generality to exclude the least element of a base-lattice from the base. This leads to the following concept:

3.4. Definition. A base-lattice L is called *spatial* if and only if $0 \neq 1$ and $0 \notin BL$. The full subcategory of **BL** with the spatial base-lattices as objects is **SBL**. Finally, the assignment $(L, G) \to L$ induces the forgetful functor $E: \mathbf{PGrL} \to \mathbf{SBL}$.

Obviously every scale-lattice is spatial. Therefore, the restriction of the adjunction of Theorem 2.5 yields an adjunction $S \dashv B: (\mathbf{SBL}, \mathbf{Set})$. Combining Lemma 3.2 and Definition 3.3 one obtains:

3.5. Proposition. *The assignments* $(X, \xi) \to GX = (SX, \tau(\xi))$ *and* $(f:(X, \xi) \to (Y, \eta)) \to (Gf: GX \to GY, \mathscr{A} \to Sf(\mathscr{A}))$ *define a full embedding* $G: \mathbf{PMer} \to \mathbf{PGrL}$ *with (co-)restriction* $G: \mathbf{Mer} \to \mathbf{GrL}$.

Conversely it is possible to endow certain subsets of (pre-)grill-lattices with a (pre-)merotopic structure. This construction is a generalization of the cluster-completion of nearness spaces [4, 5.7, 5.8].

3.6. Definition. Let (L, G) be a pregrill-lattice and $C \subset G$.

$$\xi_C = \{\mathscr{A} \subset \mathscr{P}C \mid \limsup \mathscr{A} \in G\}.$$

3.7. Proposition. *For each (pre-)grill-lattice (L, G) and each subset $C \subset G$ the collection ξ_C is a (pre-)merotopic structure on C.*

Proof. Let $\xi = \xi_C$.

(M1) $\mathscr{A} > \mathscr{B} \in \xi \Rightarrow \limsup \mathscr{A} \geq \limsup \mathscr{B} \in G \Rightarrow \mathscr{A} \in \xi$.

(M2) Let $A = \bigcap \mathscr{A} \neq \emptyset$. $A \subset C$ and $\mathscr{A} \subset \bar{A}$ imply $\limsup \mathscr{A} \geq \limsup \bar{A} = \sup A \in G$, i.e. $\mathscr{A} \in \xi$.

(M3) $\limsup \emptyset = \inf \emptyset = 1 \in G \Rightarrow \emptyset \in \xi$,
$\limsup \mathcal{P}C = \sup \emptyset = 0 \notin G \Rightarrow \mathcal{P}C \notin \xi$.

(M4) Suppose G is lattice-grill and $\mathcal{A}, \mathcal{B} \notin \xi$.
$\limsup(\mathcal{A} \vee \mathcal{B}) = \inf\{\sup(A \cup B) \mid A \in \mathcal{A}, B \in \mathcal{B}\} = \limsup \mathcal{A} \vee \limsup \mathcal{B} \in G$.
Therefore $\mathcal{A} \vee \mathcal{B} \notin \xi$. $\square$

3.8. Definition. The functor $M: \mathbf{GrL} \to \mathbf{Mer}$ ($\mathbf{PGrL} \to \mathbf{PMer}$) defined by

$$(L, G) \to (BL, \xi_{BL})$$
$$(f: (L, G) \to (K, H) \to (Bf: (BL, \xi_{BL}) \to (BK, \xi_{BK}))$$

is called *(pre-)merotopic base-functor. ML* is the *base-space* of the (pre-)grill-lattice $L = (L, G)$.

These assignments are well defined, since for each $\mathcal{A} \in \xi_{BL}$ $\limsup(f\mathcal{A}) = f(\limsup \mathcal{A}) \in f[G] \subset H$ and therefore $f\mathcal{A} \in \xi_{BK}$, i.e. Bf is uniformly continuous.

3.9. Theorem. *The functor* $G: \mathbf{PMer} \to \mathbf{PGrL}$ *is left adjoint to* $M: \mathbf{PGrL} \to \mathbf{PMer}$ *and analogously the (co-)restrictions to* **Mer** *and* **GrL**. *The unit* $\eta = (\eta_X)$ *with*

$$\eta_X: \ (X, \xi) \to MGX = (BSX, \xi_{BSX}), \quad x \to \dot{x}$$

is a natural isomorphism.

$$\begin{tikzcd} \mathbf{GrL} \arrow[r, shift left, "M"] & \mathbf{Mer} \arrow[l, shift left, "G"] \end{tikzcd}$$

$$\begin{tikzcd} \mathbf{PGrL} \arrow[r, shift left, "M"] \arrow[d, "E"'] & \mathbf{PMer} \arrow[l, shift left, "G"] \arrow[d, "V"] \\ \mathbf{SBL} \arrow[r, shift left, "B"] & \mathbf{Set} \arrow[l, shift left, "S"] \end{tikzcd}$$

Proof. Consider the following situation (cf. 2.5):

$$\begin{tikzcd} (X, \xi) \arrow[r, "\eta_X"] \arrow[dr, "f"'] & MG(X, \xi) \arrow[d, dashed, "M\bar{f}"] & G(X, \xi) \arrow[d, dashed, "\bar{f}"] \\ & M(K, H) & (K, H) \end{tikzcd}$$

If $\mathcal{A} \in \tau(\xi)$, then $f\mathcal{A} \in \xi_{BK}$ and therefore $\bar{f}(\mathcal{A}) = \limsup(f\mathcal{A}) \in BK \subset H$, i.e. $\bar{f}: G(X, \xi) \to (K, H)$ is grill-continuous. Moreover, for any $\mathcal{A} \subset \mathcal{P}X$ $\mathcal{A} = \limsup_{S(X)}(\eta_X \mathcal{A})$.

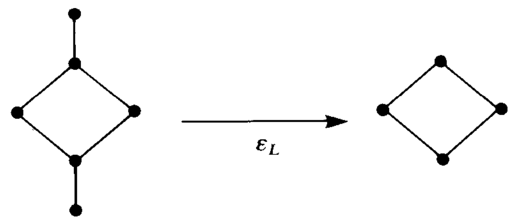
This implies $\mathcal{A} \in \xi \Leftrightarrow \eta_X \mathcal{A} \in \xi_{BSX}$, which ensures η_X to be an isomorphism in **PMer**. The statements with respect to the restrictions follow immediately from Propositions 3.5 and 3.7. $\square$

Since pregrill-lattice are generated by their bases, every uniformly continuous base-homomorphism (with respect to the associated base-spaces) is a grill-continuous map:

3.10. Proposition. *Suppose* (L, G), $(K, H) \in \mathbf{PGrL}$, $h : L \to K$ *base-homomorphism,* $g : ML \to MK$ *uniformly continuous and* $Bh = Vg$. *Then* $h : (L, G) \to (K, H)$ *is grill-continuous.*

Proof. Let $x \in G$, $x = \limsup \mathcal{A}$ with $\mathcal{A} \subset \mathcal{P}BL$. Then $\mathcal{A} \in \xi_{BL}$, hence $g\mathcal{A} \in \xi_{BK}$ and therefore $h(x) = h(\limsup \mathcal{A}) = \limsup(h\mathcal{A}) = \limsup(g\mathcal{A}) \in H$. $\square$

The set of pregrill-structures of a spatial base-lattice L is always a complete lattice with greatest element $L - \{0\}$, but this structure is in general not an indiscrete structure, i.e. initial with respect to the empty source. As a counter-example consider $L = 2^2$ with $BL = L - \{0, 1\}$, $K = SBL$ and $\varepsilon_L : K \to L$.



K L

If K is endowed with the grill-structure $H = K - \{0\}$, ε_L cannot be made into a grill-continuous function, since $\varepsilon_L[H] = L$. Therefore $E : \mathbf{PGrL} \to \mathbf{SBL}$ is not a topological functor.

3.11. Theorem. *The functor* $E : \mathbf{PGrL} \to \mathbf{SBL}$ *is mono-topological, i.e. for every E-mono-source* $(f_i : L \to E(L_i, G_i))_I$ *there exists an E-initial E-lift, which is given by*

$$(\bar{f}_i : (L, G_p) \to (L_i, G_i))_I$$

with $G_p = \bigcap \{f_i^{-1}[G_i] \mid i \in I\}$.

Proof. Consider the following diagram:

$$\begin{tikzcd} (K, H) \arrow[r, dashed, "\bar{g}"] \arrow[dr, "g_i"'] & (L, G_p) \arrow[d, "\bar{f}_i"] \\ & (L_i, G_i) \end{tikzcd} \qquad \begin{tikzcd} K \arrow[r, "g"] \arrow[dr, "Eg_i"'] & L \arrow[d, "f_i"] \\ & L_i \end{tikzcd}$$

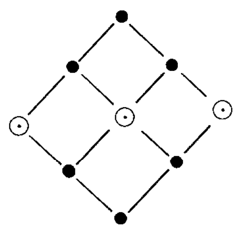
Since $(f_i)_I$ is a mono-source, I must be nonempty, because otherwise for each $M \in \mathbf{PGrL}$ $|\hom(EM, L)| \leq 1$ and therefore $|BL| = |\hom(S1, L)| \leq 1$, i.e. $L \simeq S\emptyset$ or

$L \simeq S1$. On the other hand in both cases $|\text{hom}(S2, L)| > 1$ (where $S2$ denotes the scale of a two-element-set), a contradiction. Now $I \neq \emptyset$ implies $\emptyset \neq G_p \neq L$. Clearly G_p is an upper set containing BL. In order to show that $g:(K, H) \to (L, G_p)$ is grill-continuous consider $y \in H$ and $i \in I$. Then

$$f_i(g(y)) = g_i(y) \in G_i$$

and therefore $g(y) \in G_p$. □

The restriction of E to the category of grill-lattices is however not mono-topological, since the set of all grill-structures on a spatial base-lattice does not build a complete lattice in general (e.g. $L = 3^2$ with base-elements $(0, 2)$, $(1, 1)$ and $(2, 0)$).



If one applies the results of Tholen concerning Wyler's taut lift theorem [13, 2.1], one obtains the following convenient conclusion:

3.12. Proposition. *The (pre-)merotopic base-functor preserves initial sources, i.e. if* $(f_i:(L, G) \to (L_i, G_i))_I$ *is an E-initial source in* **GrL** (**PGrL**), *then*

$$(Bf_i:(BL, \xi_{BL}) \to (BL_i, \xi_{BL_i}))_I$$

is a V-initial source in **Mer** (**PMer**).

4. Nearness-lattices

The lattice-theoretical characterization of nearness spaces certainly needs an investigation of the action of the closure operator on the associated grill-lattice.

4.1. Proposition. *Let* (X, ξ) *be a premerotopic space,* $A \subset X$ *and* $\mathscr{A} \in S(X)$.

(1) $\overline{\text{cl}_\xi A} = \sup_{S(X)}\{\dot{x} \mid \dot{x} \wedge \bar{A} \in \tau(\xi)\}$.

(2) $\text{stack}(\text{cl}_\xi \mathscr{A}) = \inf_{S(X)}\{\sup_{S(X)}\{\dot{x} \mid \dot{x} \wedge \bar{B} \in \tau(\xi)\} \mid B \in \mathscr{A}\}$.

The verification of these equations is quite elementary and therefore omitted. Now it is possible to introduce an analogue of the closure operator for arbitrary pregrill-lattices.

4.2. Definition. Suppose (L, G) is a pregrill-lattice and $x \in L$.

(1) $\text{cl}_G x = \inf\{\sup\{b \in BL \mid b \wedge \sup A \in G\} \mid A \in \mathscr{A}_x\}$.

(2) $J(L) = \{y \in L \mid \exists B \subset BL \text{ with } y = \sup B\}$ is called the *join-set* of (L, G).

4.3. Lemma. *Let (L, G) be pregrill-lattice, $x \in L$, $y \in J(L)$.*

(1) $\mathrm{cl}_G\, y = \sup\{b \in BL \mid b \wedge y \in G\}$.

(2) $\mathrm{cl}_G\, x = \inf\{\mathrm{cl}_G(\sup A) \mid A \in \mathscr{A}_x\}$.

Proof. (1): Let $B \subset BL$ and $y = \sup B$. Then $B \in \mathscr{A}_y$ and $\mathrm{cl}_G\, y \leqslant \sup\{b \in BL \mid b \wedge y \in G\} = z$. Moreover, for every $A \in \mathscr{A}_y$ $\{b \in BL \mid b \wedge y \in G\} \subset \{b \in BL \mid b \wedge \sup A \in G\}$. Therefore $z \leqslant \mathrm{cl}_G\, y$.

(2): Apply (1). □

4.4. Lemma. *Let (L, G) pregrill-lattice, $A \subset BL$, $\mathscr{A} \subset \mathscr{P}BL$.*

(1) $\mathrm{cl}_G(\sup A) = \sup(\mathrm{cl}_{\xi_{BL}}(A))$.

(2) $\mathrm{cl}_G(\limsup \mathscr{A}) \leqslant \limsup(\mathrm{cl}_{\xi_{BL}}(\mathscr{A}))$.

Proof. (1) Apply Lemma 4.3(1).

(2) Let $x = \limsup \mathscr{A}$. Then $\mathscr{A}_x \supset \mathscr{A}$ and the assertion follows from (1). □

The inequality in Lemma 4.4(2) is strict in general, for instance in case of the largest grill-structure on 3^2 with its $\wedge$-primes as a base.

4.5. Definition. A grill-lattice (L, G) is called *nearness-lattice* if and only if

$$\forall A \subset J(L) \quad (\inf A \notin G \Rightarrow \inf\{\mathrm{cl}_G\, a \mid a \in A\} \notin G).$$

The category of nearness-lattices and grill-continuous maps is denoted by **NeL**.

4.6. Lemma. *Suppose (L, G) is a nearness-lattice and $x \in L$.*

$$\mathrm{cl}_G\, x \in G \;\Rightarrow\; x \in G.$$

Proof. Insert $A = \{\sup C \mid C \in \mathscr{A}_x\}$ in the defining relation of Definition 4.5. Then the assertion follows from Lemma 4.3(2). □

In view of these properties it emerges that the nearness-lattices are precisely the lattice-theoretical counterparts of the nearness spaces.

4.7. Theorem. (1) *A merotopic space (X, ξ) is a nearness space if and only if $G(X, \xi)$ is a nearness-lattice.*

(2) *A grill-lattice (L, G) is a nearness-lattice if and only if its base-space $M(L, G)$ is a nearness space.*

(3) $M:\mathbf{NeL}\to\mathbf{Near}$ *is right adjoint to* $G:\mathbf{Near}\to\mathbf{NeL}$. *In particular every nearness space is (isomorphic to) the base space of a nearness-lattice.*

$$\begin{tikzcd} \mathbf{NeL} \arrow[r, shift left, "M"] \arrow[d] & \mathbf{Near} \arrow[l, shift left, "G"] \arrow[d] \\ \mathbf{GrL} \arrow[r, shift left, "M"] & \mathbf{Mer} \arrow[l, shift left, "G"] \end{tikzcd}$$

In [14] several subcategories of merotopic spaces (e.g. grill-determined spaces, topological nearness spaces and contigual spaces) are investigated in a similar manner.

The scale of a (uniform) nearness space is different from the 'scale of a uniform space', which has been investigated by Kent for the purpose of a lattice-theoretical representation of the Weil-completion [8] and of the Samuel-compactification [9] of a uniform space. The main difference lies in the fact that he only considers (an equivalent of) the near stacks, whereas in the present representation also the entire family of all stacks plays an important role and enables us to represent not only the complete but all nearness spaces.

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